Models of Theoretical Physics

Baiesi's Exercises

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Lesson 1

Exercise 1.1 (Multivariate Gaussian Integral)**:**

Given $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{b} = (1, 0)^T$ and the matrix A:

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

compute the Gaussian integrals:

$$Z(A) = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{x} \exp\left(-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}\right)$$

$$Z(A, \boldsymbol{b}) = \int_{\mathbb{R}^2} \mathrm{d}^2\boldsymbol{x} \exp\left(-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b} \cdot \boldsymbol{x}\right)$$

Solution. We use the following formulas:

$$Z(A) = \sqrt{\frac{(2\pi)^n}{\det(A)}}$$

$$Z(A, \boldsymbol{b}) = \sqrt{\frac{(2\pi)^n}{\det(A)}} \exp\left(\frac{1}{2}\boldsymbol{b} \cdot (A^{-1}\boldsymbol{b})\right)$$

Note that $\det A = 8$, and:

$$A^{-1} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Then:

$$Z(A, 0) = \frac{(2\pi)^{2/2}}{\sqrt{8}} = \frac{\pi}{\sqrt{2}}$$

$$\frac{1}{2}\boldsymbol{b} \cdot (A^{-1}\boldsymbol{b}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{16}$$

$$Z(A, \boldsymbol{b}) = \frac{\pi}{\sqrt{2}} \exp\left(\frac{3}{16}\right)$$

Exercise 1.2 (Steepest Descent Approximation)**:**

With the saddle-point strategy, compute the approximation for large s of the following integral:

$$I(s) = \int_{-\infty}^{\infty} e^{sx - \cosh x} dx \tag{1.1}$$

Solution. Since the integral is over the real line, we can use Laplace's formula. Let $f(x)$ be a twice-differentiable function with a unique global maximum at $x_0 \in (a, b)$. Then:

$$\int_a^b e^{nf(x)}\, \mathrm{d}x \underset{n\to\infty}{\approx} \sqrt{\frac{2\pi}{n|f''(x_0)|}} e^{nf(x_0)} \tag{1.2}$$

This comes by expanding f to second order about the maximum:

$$f(x) \approx f(x_0) - \frac{1}{2}|f''(x_0)|(x - x_0)^2$$

So that:

$$\begin{aligned} \int_a^b e^{nf(x)}\, \mathrm{d}x &\approx e^{nf(x_0)} \int_a^b \exp\left(-\frac{1}{2}n|f''(x_0)|(x - x_0)^2\right) \\ &\underset{n\to\infty}{\approx} e^{nf(x_0)} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}n|f''(x_0)|(x - x_0)^2\right) \end{aligned}$$

Because x_0 is not an end-point, for $n \to \infty$ the gaussian becomes very "peaked" inside (a, b), allowing to compute its integral as if it was on $\mathbb{R}$. Then, computing the gaussian integral leads back to (1.2).

In our case we start by collecting a s in the exponential argument:

$$I(s) = \int_{-\infty}^{\infty} \exp\left(s \underbrace{\left(x - \frac{\cosh x}{s}\right)}_{f(x)}\right)$$

Now $I(s)$ is in the form needed for (1.2). We find the maximum of $f(x)$ by differentiating:

$$\begin{aligned} f'(x) &= 1 - \frac{\sinh x}{s} \overset{!}{=} 0 \Rightarrow x_0 = \sinh^{-1} s \\ f''(x) &= -\frac{\cosh x}{s} \Rightarrow f''(x_0) = -\frac{\cosh \sinh^{-1} s}{s} = -\frac{\sqrt{1+s^2}}{s} < 0 \end{aligned}$$

Finally, by applying (1.2) we obtain the result:

$$\begin{aligned} I(s) &\underset{s\to\infty}{\approx} \sqrt{\frac{2\pi}{s}} \sqrt{\frac{s}{\sqrt{1+s^2}}} \exp\left(s \sinh^{-1} s - \cosh \sinh^{-1} s\right) = \\ &= \frac{\sqrt{2\pi}}{(1+s^2)^{1/4}} \exp\left(s \sinh^{-1} s - \sqrt{1+s^2}\right) \end{aligned}$$

Exercise 1.3 (Laplace's formula)**:**

With the saddle-point strategy, compute the approximation for large N of the following integral:

$$I(N) = \int_0^\infty \underbrace{\cos(x)}_{g(x)} \exp\Big(-N \underbrace{\Big[\Big(x-\frac{\pi}{3}\Big)^2 + \Big(x-\frac{\pi}{3}\Big)^4\Big]\Big)}_{f(x)} dx$$

Solution.
For this exercise we can use Laplace's formula (1.2) with:

$$f(x) = -\left[\left(x-\frac{\pi}{3}\right)^2 + \left(x-\frac{\pi}{3}\right)^4\right]$$

As this follows by approximating the integral with its most important value at the *maximum*, the exponential prefactor $g(x)$ will appear as a prefactor of the solution evaluated at the maximum x_0: $g(x_0)$.

By looking at $f(x)$ one can see directly that it has a global maximum in $x_0 = \pi/3$. In fact:

$$f'(x) = -\left[2\left(x-\frac{\pi}{3}\right) + 4\left(x-\frac{\pi}{3}\right)^3\right] \stackrel{!}{=} 0 \Leftrightarrow x_0 = \frac{\pi}{3}$$
$$f''(x) = -\left[2 + 12\left(x-\frac{\pi}{3}\right)^2\right] \Rightarrow f''(x_0) = -2 < 0$$

And so we arrive at:

$$I(N) \underset{N\to\infty}{\approx} \cos(\pi/3)\sqrt{\frac{2\pi}{N|-2|}} = \frac{1}{2}\sqrt{\frac{\pi}{N}}$$

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Lesson 2

Exercise 2.1 (Fourier transform of derivative)**:**

Show that the following formula holds for the Fourier transform ($\mathcal{F}(f) = \tilde{f}(k)$) of a derivative of the function $f(x)$ (under the usual mathematical assumptions for having a Fourier transform and its derivative):

$$\mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{d}x}\theta(x)\right) = ik\tilde{f}(k)$$

Solution.

$$\mathcal{F}\left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right](k) = \int_{\mathbb{R}} \mathrm{d}x\,(\partial_x f(x))e^{-ikx} \underset{(a)}{=} \cancel{e^{ikx} f(x)\Big|_{x=-\infty}^{x=+\infty}} + ik\int_{\mathbb{R}} \mathrm{d}x\, e^{ikx} f(x) = ik\tilde{f}(k)$$

where in (a) we performed an integration by parts. The boundary term vanishes because we assume $f, f' \in L^2(\mathbb{R})$ to be able to compute their Fourier transform, so that $f(x) \to 0$ for $|x| \to \infty$.

Exercise 2.2 (Fourier transform of **1**)**:**

Show that $\mathcal{F}(1) = 2\pi\delta(k)$.

Solution. By applying the definition of the $\delta(k)$ distribution:

$$\mathbb{F}[1](k) = 2\pi \underbrace{\int_{\mathbb{R}} \mathrm{d}x\,\frac{e^{-ikx}}{2\pi}}_{\delta(k)} = 2\pi\delta(k)$$

Alternatively, we can show that:

$$\mathcal{F}^{-1}[2\pi\delta(k)](x) = \int_{\mathbb{R}} \frac{2\pi}{2\pi} e^{ikx}\delta(k)\,\mathrm{d}k \underset{(a)}{=} e^{i0x} = 1$$

where in (a) we applied $\langle \delta, f \rangle = f(0)$.

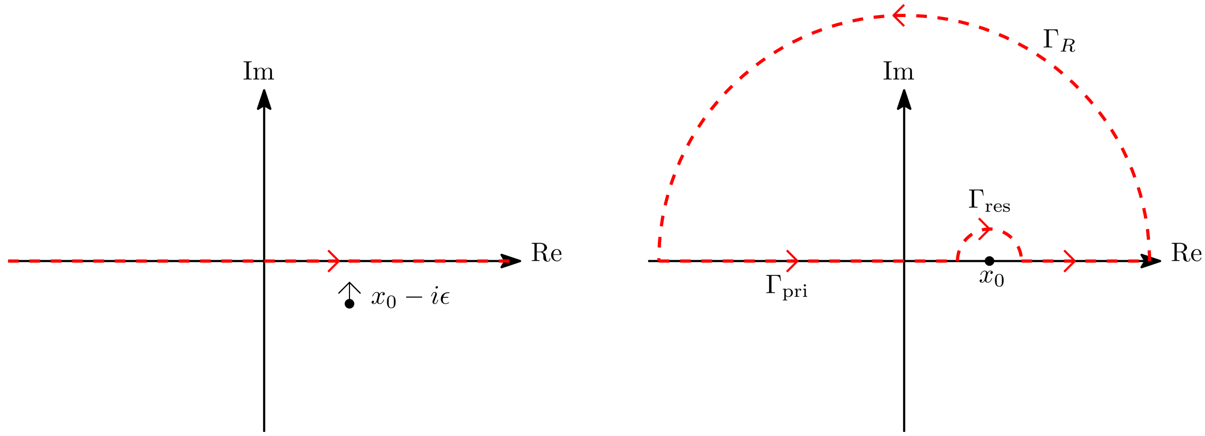

Figure (2.1) – **Left**: integral on the real line with approaching singularity. **Right**: integral using a closed curve and a shifted singularity.

Exercise 2.3 (Prescription $\boldsymbol{i\epsilon}$)**:**

To complete the case discussed during the lecture, compute:

$$\lim_{\epsilon \to 0} \frac{1}{x - x_0 + i\epsilon} = P\left[\frac{1}{x - x_0}\right] - i\pi\delta(x - x_0)$$

Note that this limit and that discussed in the lecture are a physicists' crude shorthand notation for the full equation:

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{f(x)}{x - x_0 \mp i\epsilon} \, \mathrm{d}x = P \int_{-\infty}^{+\infty} \frac{f(x)}{x - x_0} \, \mathrm{d}x \pm i\pi f(x_0)$$

and $f(z) \to 0$ for $|z| \to \infty$ and analytic in the $\mathrm{Im}(z) \geq 0$ portion of the complex plane.

Solution. The integral on the real line with an approaching singularity from $\mathrm{Im}(z) \leq 0$ (figure 2.1, left) can be computed by using the closed curve shown in (figure 2.1, right), and applying Cauchy's integral theorem. The integrand, extended to the complex plane, is:

$$g(z) = \frac{f(z)}{z - (x_0 - i\epsilon)}$$

By hypothesis, the integral over Γ_R vanishes:

$$\int_{\Gamma_R} g(z) \, \mathrm{d}z = 0$$

Then, the integral over Γ_{pri} is, by definition, the principal part of the real integral:

$$\int_{\Gamma_{\text{pri}}} g(z) \, \mathrm{d}z = P \int_{\mathbb{R}} \mathrm{d}x \, \frac{f(x)}{x - x_0}$$

And finally, the integral over Γ_{res} is equal to half the residue at x_0, with a minus sign given by the clockwise rotation:

$$\int_{\Gamma_{\text{res}}} g(z) \, \mathrm{d}z = -\frac{2\pi i}{2} f(x_0) = -\pi i f(x_0)$$

This proves the required relation:

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\frac{f(x)}{x-x_0+i\epsilon}\,\mathrm{d}x = P\int_{\mathbb{R}}\mathrm{d}x\,\frac{f(x)}{x-x_0} - i\pi f(x_0)$$

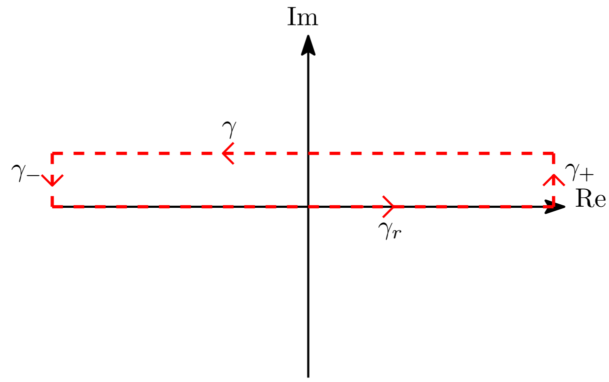

Figure (2.2) – Closed path for the gaussian integral.

Exercise 2.4 (Gaussian integral)**:**

Compute the Gaussian integral

$$I = \int_{-\infty}^{\infty}\mathrm{d}x\exp\left(-ax^2+bx\right) = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right)$$

for $a\in\mathbb{R}, a>0$ and complex $b=\beta+i\nu$ (with $\beta,\nu\in\mathbb{R}$). For the solution, one may shift to a new variable z with $x=z+iq$, so that the exponent in the integral does not contain a term $\sim iz$ and the new path of integration can be mapped back to the real axis by using Cauchy's theorem.

Solution. The starting integral is:

$$I = \int_{\mathbb{R}}\mathrm{d}x\exp\left(-ax^2+\beta x+i\nu x\right)$$

We then perform a change of variables:

$$x = z+iq \Rightarrow \mathrm{d}x = \mathrm{d}z$$

moving the integral from the real line to γ, i.e. the horizontal line at $\mathrm{Im}\, z = iq$.

$$I = \int_{\gamma}\mathrm{d}z\exp\left(-a(z+iq)^2+\beta(z+iq)+i\nu(z+iq)\right)$$

Expanding the exponential argument leads to:

$$\begin{aligned}-a(z^2-q^2+2iqz)+\beta z+i\beta q+i\nu z-\nu q &=\\ -az^2+z\beta+iz(\nu-2qa)+aq^2-\nu q+i\beta q&\end{aligned}$$

To remove the iz term we set $\nu - 2qa = 0 \Rightarrow q = \nu/(2a)$, leading to:

$$= -az^2 + z\beta + \frac{a\nu^2}{4a^2} - \frac{\nu^2}{2a} + i\frac{\beta\nu}{2a} = -az^2 + z\beta - \frac{\nu^2}{4a} + i\frac{\beta\nu}{2a}$$

Substituting back in the integral:

$$I = \int_\gamma \mathrm{d}z \exp\left(-az^2 + z\beta\right) \exp\left(-\frac{\nu^2}{2a} + i\frac{\beta\nu}{2a}\right)$$

Consider now the closed path shown in fig. 2.2. In the limit where γ_r goes from $-\infty$ to $+\infty$, the integrals over γ_+ and γ_- vanish, as $\exp\left(-az^2 + bz\right) \to 0$ for $|z| \to \infty$. Then, as the closed path does not contain any singularity, by Cauchy's integral theorem we have that the integral along γ is the same as the integral on the real line (assuming the same orientation). This allows us to evaluate I on the real line:

$$\begin{aligned} I &= \int_{\mathbb{R}} \mathrm{d}x \exp\left(-ax^2 + x\beta\right) \exp\left(-\frac{\nu^2}{2a} + i\frac{\beta\nu}{2a}\right) = \\ &= \sqrt{\frac{\pi}{a}} \exp\left(\frac{\beta^2}{4a} - \frac{\nu^2}{4a} + i\frac{\beta\nu}{2a}\right) = \\ &= \sqrt{\frac{\pi}{a}} \exp\left(\frac{(\beta + i\nu)^2}{4a}\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \end{aligned}$$

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Lesson 3

Exercise 3.1 (Cauchy distribution)**:**

Expand the details of these passages:

$$P(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{-x^*|k|+ikx} = \frac{1}{\pi}\int_{0}^{\infty} \mathrm{d}k\, e^{-x^* k}\cos kx = \frac{1}{\pi x^*}\frac{1}{1+\left(\frac{x}{x^*}\right)^2}$$

used to find the one-dimensional Cauchy distribution. Finding the last term by skipping entirely the $\cos kx$ step is also an option. Here $x = 0$ at $t = 0$ and $x^* = D_1 t$.

Solution. We start from the *generalized diffusion equation*:

$$\begin{cases} \partial_t P(x,t) = D_\mu \frac{\partial}{\partial |x|^\mu} P(x,t) \\ P(x,0) = \rho(x) \end{cases}$$

with $0 < \mu < 2$. The *fractional* derivative makes sense after passing in Fourier space:

$$\partial_t \tilde{P}(k,t) = -D_\mu |k|^\mu \tilde{P}(k,t) \Leftrightarrow \partial_t [\exp(D_\mu |k|^\mu t)\tilde{P}(k,t)] = 0$$

This means that the exponential must be time independent:

$$\tilde{f}(k) \equiv \exp(D_\mu |k|^\mu t)\tilde{P}(k,t) \Rightarrow \tilde{P}(k,t) = \tilde{f}(k)\exp(-D_\mu |k|^\mu t)$$

Since $\tilde{f}(k)$ so defined does not depend on time, we can compute it by setting $t = 0$, leading to $\tilde{f}(k) = \tilde{P}(k,0) = \tilde{\rho}(k)$.
Cauchy random flights are found by setting $\mu = 1$. The equation becomes:

$$\tilde{P}_C(k,t) = \tilde{\rho}(k)\exp(-D_1|k|t)$$

Assuming that the particle is localized in $x = 0$ at $t = 0$, then $\rho(x) = \delta(x)$, and so $\tilde{\rho}(k) = \mathbb{F}[\delta(x)](k) = 1$, leading to:

$$\tilde{P}_C(k,t) = e^{-D_1|k|t} = \exp(-x^*(t)|k|)$$

To return to position space, we compute a Fourier anti-transform:

$$P_C(x,t) = \mathcal{F}^{-1}[\tilde{P}_C](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k \exp\left(-x^*(t)|k| - ikx\right) =$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, e^{-x^*(t)|k|}\left[\cos(-kx) + i\sin(-kx)\right]$$

Note that the domain is symmetric, and the red terms are even, while the blue one is odd. So the sin contribution will be 0, leading to:

$$P_C(x,t) = \frac{1}{2\pi} 2\int_0^{+\infty} \mathrm{d}k\, e^{-x^*(t)k}\cos(kx)$$

The integral can be computed with a double integration by parts:

$$I = \int_0^{+\infty} e^{-x^*k}\cos(kx)\,\mathrm{d}k = -\cos(kx)\frac{1}{x^*}e^{-x^*k}\Big|_{k=0}^{k=+\infty} + x\sin(kx)(x^*)^{-2}e^{-x^*k}\Big|_{k=0}^{k=+\infty}$$
$$-\int_0^{+\infty}\mathrm{d}k\, x^2\cos(kx)(x^*)^{-2}e^{-x^*k} =$$
$$= \frac{1}{x^*} - \frac{x^2}{(x^*)^2}I \Rightarrow I = \frac{1}{x^*}\frac{1}{1+\left(\frac{x}{x^*}\right)^2}$$

And readding the $1/\pi$ factor leads to the desired solution:

$$P_C(x,t) = \frac{1}{\pi x^*}\frac{1}{1+\left(\frac{x}{x^*}\right)^2}$$

Exercise 3.2 (Transition probabilities and Cauchy flights)**:**

With the Cauchy jump distribution with typical displacement $x^* = D_1 t$ at time t (see previous exercise, setting $x =$ displacement), compute the probability $P(x,t)$ to find the particle at position x at time t for such a Levy process, when the initial distribution is uniform and bound as $P(x,0) = \rho(x) = 1/(2a)$ for $x \in [-a,a]$ and $\rho(x) = 0$ otherwise.

Solution. The initial distribution is given by:

$$P(x,0) = \rho(x) = \begin{cases} \frac{1}{2a} & x \in [-a,+a] \\ 0 & \text{otherwise} \end{cases}$$

The probability of a particle being in x at t is obtained by *propagating* the initial distribution:

$$\begin{aligned}
P(x,t) &= \int_{\mathbb{R}} \mathrm{d}x_0 \, P(x,t|x_0,0)P(x_0,0) = \\
&= \int_{-a}^{+a} \mathrm{d}x_0 \, \frac{1}{\pi x*} \frac{1}{1+\left(\frac{x-x_0}{x^*}\right)^2} \frac{1}{2a} = \\
&= -\frac{1}{2a\pi x^*(t)} \arctan\left(\frac{x-x_0}{x^*}\right)\Big|_{x_0=-a}^{x_0=+a} = \\
&= \frac{1}{2\pi a x^*(t)} \left[-\arctan\left(\frac{x-a}{x^*}\right) + \arctan\left(\frac{x+a}{x^*}\right)\right]
\end{aligned}$$

Exercise 3.3 (Numerical simulation - optional)**:**

Check numerically that the sum $S_n = x_1 + \ldots + x_n$ of n i.i.d. variables $x \in \mathbb{R}$, each one distributed according to

$$p(x) = \frac{1}{4x^2} \quad \text{for } |x| > 1, \quad p(x) = 1/4 \text{ for } |x| \leq 1$$

converges to a Cauchy distribution

$$P_{\text{Cauchy}}(Y) = \frac{1}{\pi\left(1+Y^2\right)}$$

after a suitable rescaling $Y_n = \gamma S_n / n^\beta$. What are γ and β?

See the Jupyter notebook at this link: `https://github.com/Einlar/data_notes/blob/revision/Models/Plots/Baiesi3_3-simulation.ipynb`.

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Lesson 4

Consider the two-state model with states at position $x_1 = -c$ and $x_2 = +c$ and probability p to be in state $-c$, which evolves according to:

$$\dot{p} = -Wp + \frac{W}{2} + \epsilon \sin(\omega_s t)$$

Exercise 4.1:

For $\epsilon = 0$, show that the correlation time function is:

$$C(t) = \langle x(t)x(0) \rangle = c^2 e^{-W|t|}$$

Solution. The evolution equation for $\epsilon = 0$ reads:

$$\dot{p} = -Wp + \frac{W}{2} = -W\left(p - \frac{1}{2}\right) = -W\Delta p$$

With $\Delta p = p - 1/2$. As $\dot{\Delta p} = \dot{p}$ we get an equivalent ODE that can be solved by separation of variables:

$$\dot{\Delta p} = -W\Delta p \Rightarrow \Delta p(t) = \Delta p(0) e^{-Wt}$$

Substituting back:

$$p(t) - \frac{1}{2} = \left(p(0) - \frac{1}{2}\right) e^{-Wt} \Rightarrow p(t) = \left(p(0) - \frac{1}{2}\right) e^{-Wt} + \frac{1}{2}$$

and:

$$1 - p(t) = \frac{1}{2} - \left(p(0) - \frac{1}{2}\right) e^{-Wt}$$

We can now compute the correlator:

$$\langle x(t)x(0) \rangle = \int_{\mathbb{R}^2} \mathrm{d}x\, x\mathbb{P}(x,t)x\mathbb{P}(x,0) = \int_{\mathbb{R}^2} x^2 \mathbb{P}(x,t)\mathbb{P}(x,0) \qquad t > 0$$

There are only two possible values for x: c and $-c$. $p(t)$ is the probability of $x_t = -c$, i.e. $P(-c, t)$. By conservation of probability:

$$\mathbb{P}(c,t) = 1 - \mathbb{P}(-c,t) = 1 - p(t)$$

Substituting in the expression for the correlator:

$$\begin{aligned}\langle x(t)x(0)\rangle = {} & (-c)^2 p(t)p(0) + c^2(1-p(t))(1-p(0)) + \\ & + c(-c)p(t)(1-p(0)) + (-c)c(1-p(t))p(0)\end{aligned}$$

For simplicity of notation, let:

$$p(t) \equiv p_t; \qquad p(0) \equiv p_0; \qquad p_t = \left(p_0 - \frac{1}{2}\right) A + \frac{1}{2}; \qquad A \equiv e^{-Wt}$$

Then:

$$\begin{aligned}\langle x(t)x(0)\rangle &= c^2[p_t p_0 + (1-p_t)(1-p_0) - p_t(1-p_0) - p_0(1-p_t)] = \\ &= c^2[\,p_t p_0 + 1 + p_t p_0 - p_t - p_0 - p_t + p_0 p_t - p_0 + p_0 p_t\,] = \\ &= c^2[\,4p_t p_0 - 2p_t - 2p_0 + 1] = \\ &= c^2[4p_0^2 A + 4p_0(-A/2) + 4p_0/2 - 2p_0 A + A - 1 - 2p_0 + 1] = \\ &= c^2[4p_0^2 A - 2p_0 A + 2p_0 - 2p_0 A + A - 2p_0] = \\ &= c^2[4p_0^2 A - 4p_0 A + A] = e^{-Wt} c^2(4p_0^2 - 4p_0 + 1) = \\ &= e^{-Wt} c^2 (2p_0 - 1)^2\end{aligned}$$

For $p_0 = 1$ (system initially in $-c$):

$$\langle x(t)x(0)\rangle = c^2 e^{-Wt} \qquad t > 0$$

The same argument works for $t < 0$, with the only difference being a sign. So, in the general case:

$$\langle x(t)x(0)\rangle = c^2 e^{-W|t|}$$

Exercise 4.2:

For $\epsilon = 0$, use the Wiener-Kintchine Theorem:

$$P(\omega) = 4\int_0^{\infty} C(t)\cos(\omega t)\,\mathrm{d}\omega \tag{4.1}$$

tho show that the power spectrum in this case is:

$$P^{(0)}(\omega) = 4c^2 \frac{W}{W^2+\omega^2}$$

Solution. For $\epsilon = 0$ we derived in the previous exercise that:

$$C(t) = \langle x(t)x(0)\rangle = c^2 e^{-W|t|}$$

Inserting in the Wiener-Kintchine theorem (4.1):

$$\begin{aligned}
P(\omega) = 4\int_0^{+\infty} c^2 e^{-W|t|}\cos(\omega t)\,\mathrm{d}t =\\
&= 4c^2\int_0^{+\infty} e^{-Wt}\cos(\omega t)\,\mathrm{d}t =\\
&= 2c^2\int_0^{+\infty} e^{-Wt}(e^{i\omega t} - e^{-i\omega t})\,\mathrm{d}t =\\
&= 2c^2\int_0^{+\infty} [e^{it(\omega - W/i)} - e^{-it(\omega + W/i)}]\,\mathrm{d}t =\\
&= 2c^2\int_0^{+\infty} [e^{it(\omega + iW)} - e^{-it(\omega - iW)}]\,\mathrm{d}t =\\
&\underset{(a)}{=} 2c^2\left[-\frac{1}{i\omega - W} + \frac{1}{i\omega + W}\right] = 2c^2\left[\frac{1}{W - i\omega} + \frac{1}{W + i\omega}\right] =\\
&= 2c^2\left[\frac{W + \cancel{i\omega} + W - \cancel{i\omega}}{W^2+\omega^2}\right] = 4c^2\frac{W}{W^2+\omega^2}
\end{aligned}$$

To compute the integral in (a) we used the following Fourier transform:

$$\begin{aligned}
\int_0^{+\infty} e^{-it(\omega - i\omega_0)}\,\mathrm{d}t = \int_{\mathbb{R}} \theta(t)e^{-it(\omega - i\omega_0)} = \tilde{\theta}(\omega - i\omega_0) = \frac{1}{i(\omega - i\omega_0)} =\\
= \frac{1}{i\omega + \omega_0}
\end{aligned}$$

Exercise 4.3:

For $\epsilon \neq 0$, show that the *signal-to-noise ratio* is maximum at $\kappa^* = \Delta V$ if the rates follow the Kramers formula:

$$W_{1,2} = \exp\left[-\frac{2\Delta V}{\kappa} \mp \frac{2V_1}{\kappa}\sin(\omega_s t)\right] = \frac{W}{2}\exp\left[\mp\frac{2V_1}{\kappa}\sin(\omega_s t)\right] \tag{4.2}$$

with $V_1 \ll \Delta V$ and using the correct identification for ϵ in this case.

Solution. In the *reduced* model we started from:

$$W_{1,2} = \frac{W}{2} \mp \epsilon\sin(\omega_s t) = \frac{W}{2}\left(1 \mp \frac{2\epsilon}{W}\sin(\omega_s t)\right) \tag{4.3}$$

We confront this expression with the one from (4.2), where we expand the exponential in the limit $V_1/\Delta V \ll 1$ (as the sinusoidal term is just a *perturbation*):

$$W_{1,2} \approx \exp\left(-\frac{2\Delta V}{\kappa}\right)\left(1 \mp \frac{2V_1}{\kappa}\sin(\omega_s t)\right)$$

By comparison:

$$\frac{W}{2} \sim \exp\left(-\frac{2}{k}\Delta V\right); \qquad \frac{2\epsilon}{W} \sim \frac{2V_1}{\kappa} \Rightarrow \epsilon \sim W\frac{V_1}{\kappa} \tag{4.4}$$

From previous calculations, we arrived at the following expression for the *signal to noise ratio*:

$$\mathrm{SNR}_{\omega_s} \sim \frac{\pi^2\epsilon^2}{W}$$

Substituting in the right terms (4.4) and ignoring all prefactors (as we are interested just in the position of the maximum), we arrive to:

$$\mathrm{SNR}_{\omega_s} \sim \frac{1}{k^2}\exp\left(-\frac{2}{k}\Delta V\right)$$

To find its maximum, we set its derivative with respect to k to 0:

$$\begin{aligned}
&-\frac{2}{k^3}\exp\left(-\frac{2}{k}\Delta V\right) + \frac{1}{k^2}\frac{2\Delta V}{k^2}\exp\left(-\frac{2}{k}\Delta V\right) \overset{!}{=} 0 \\
&\Rightarrow -\frac{2}{k^3}\exp\left(-\frac{2}{k}\Delta V\right) + \frac{2\Delta V}{k^4}\exp\left(-\frac{2}{k}\Delta V\right) \overset{!}{=} 0 \\
&\Rightarrow \exp\left(-\frac{2}{k}\Delta V\right)\left[-1 + \frac{\Delta V}{k}\right] = 0 \Rightarrow \frac{\Delta V}{k} = 1 \Rightarrow k = \Delta V
\end{aligned}$$

5 Lessons 5-6

Exercise 5.1:

Consider the Random Field Ising Model (RFIM), in which the disorder has variance δ^2. Proceed to arrive at the formula where the number n of replicas appears explicitly in the magnetization m:

$$m = \frac{1}{Z_1(m)} \int_{\mathbb{R}} \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(\frac{1}{2}\nu^2 + n \ln 2 \cosh(2\beta J m + \beta\delta\nu)\right) \tanh(2\beta J m + \beta\delta\nu)$$

Solution. We start from the expression for $\overline{Z^n}$ after the 2 Hubbard-Stratonovich transformations:

$$\overline{Z^m} = \left(\frac{N}{2\pi}\right)^{n/2} \left[\int_{\mathbb{R}} \mathrm{d}x \exp\left(N\left[-\frac{1}{2}nx^2 + \log Z_1(x)\right]\right)\right]^n \tag{5.1}$$

This integral is evaluated in the saddle-point approximation. Minimizing the exponential argument leads to:

$$\frac{\partial}{\partial x}\left(-\frac{1}{2}nx^2 + \log Z_1(x)\right) = 0 \Rightarrow nx = \frac{\partial}{\partial x} \log Z_1(x) \tag{5.2}$$

Recall that we define the magnetization m as:

$$\frac{x}{\sqrt{2\beta J}} = m$$

So we can change variables in (5.1) through (5.2). In particular, note that:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial m}\frac{\partial m}{\partial x} = \frac{1}{\sqrt{2\beta J}}\frac{\partial}{\partial m}$$

And so (5.1) becomes:

$$nm\sqrt{2\beta J} = \frac{1}{\sqrt{2\beta J}}\frac{\partial}{\partial m} \log Z_1(m) \Rightarrow m = \frac{1}{n}\frac{1}{2\beta J}\frac{\partial}{\partial m} \log Z_1(m) \tag{5.3}$$

We have already found that:

$$Z_1(m) = \int_{\mathbb{R}} \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + n\log[2\cosh(2\beta Jm + \beta\delta\nu)]\right)$$

Substituting in (5.3):

$$\begin{aligned} m &= \frac{1}{n}\frac{1}{2\beta J}\frac{\partial}{\partial m}\log Z_1(m) = \\ &= \frac{1}{n}\frac{1}{2\beta J}\frac{1}{Z_1(m)}\frac{\partial}{\partial m}\int_{\mathbb{R}} \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + n\log[2\cosh(2\beta Jm + \beta\delta\nu)]\right) = \\ &= \frac{1}{\cancel{n}}\frac{1}{\cancel{2\beta J}}\frac{1}{Z_1(m)}\int_{\mathbb{R}} \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + n\log[2\cosh(2\beta Jm + \beta\delta\nu)]\right)\cdot \\ &\quad \cdot \frac{\cancel{n}}{\cancel{2}\cosh(2\beta Jm + \delta\beta\nu)}\cancel{2}\sinh(2\beta Jm + \delta\beta\nu)\cancel{2\beta J} = \\ &= \frac{1}{Z_1(m)}\int_{\mathbb{R}} \frac{\mathrm{d}\nu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\nu^2 + n\log[2\cosh(2\beta Jm + \delta\beta\nu)]\right)\cdot \\ &\quad \cdot \tanh(2\beta Jm + \delta\beta\nu) \end{aligned}$$

Exercise 5.2:

With the self-consistent solution $m_{\mathrm{SC}}(m) = m$ of the RFIM, by using the condition $\partial m_{\mathrm{SC}}/\partial m = 1$ for the critical point, show that the phase transition between paramagnetic phase and ferromagnetic phase takes place where this condition is satisfied:

$$2\beta J \int_{\mathbb{R}} \mathrm{d}h\, p(h) \frac{1}{[\cosh(\beta h)]^2} = 1 \tag{5.4}$$

Solution. The self-consistent equation for the RFIM is:

$$m = \overline{\tanh(\beta[2Jm + h])}$$

Criticality is reached when the lhs and rhs are *tangent* at the origin, meaning that:

$$\frac{\partial}{\partial m}\overline{\tanh(\beta[2Jm + h])}\Big|_{m=0} \overset{!}{=} 1$$

Expanding the average leads to:

$$\begin{gathered} \int_{\mathbb{R}} \mathrm{d}h\, p(h) \frac{\partial}{\partial m}\tanh(2\beta Jm + \beta h)\Big|_{m=0} = \\ = \int_{\mathbb{R}} \mathrm{d}h\, p(h) \frac{1}{\cosh^2(2\beta Jm + \beta h)}\Big|_{m=0} (2\beta J) = 2\beta J \int_{\mathbb{R}} \mathrm{d}h\, p(h) \frac{1}{\cosh^2(\beta h)} \overset{!}{=} 1 \end{gathered}$$

Exercise 5.3:

Show that at zero temperature in the RFIM there is a disorder-driven para-ferromagnetic transition where the random field standard deviation δ and the coupling J satisfy $2J/\delta = \sqrt{\pi/2}$. For simplicity one may take $\delta = 1$.

Solution. We start from the criticality condition (5.4), inserting the distribution $p(h)$:

$$1 = 2\beta J \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\delta^2}} \exp\left(-\frac{h^2}{2\delta^2}\right) \frac{1}{\cosh^2(\beta h)} \, \mathrm{d}h$$

We introduce *reduced dimensionless variables*:

$$J' = \frac{J}{\delta}; \qquad \beta' = \beta\delta; \qquad \tilde{h} = \beta h \Rightarrow \mathrm{d}\tilde{h} = \beta \, \mathrm{d}h$$

leading to:

$$1 = 2\cancel{\beta} J' \cancel{\delta} \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{h}}{\cancel{\beta\delta}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tilde{h}^2}{2\beta'^2}\right) \frac{1}{\cosh^2(\tilde{h})}$$

In the low temperature limit $\beta \to \infty$ the exponential tends to unity:

$$2J' \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{h}}{\sqrt{2\pi}} \frac{1}{(\cosh\tilde{h})^2} = 1 \tag{5.5}$$

Note that:

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{h}} \tanh(\tilde{h}) = \frac{1}{(\cosh\tilde{h})^2}$$

And so we can evaluate (5.5):

$$\frac{2J'}{\sqrt{2\pi}} \tanh\tilde{h}\Big|_{-\infty}^{+\infty} = \frac{2J'}{\sqrt{2\pi}}(1-(-1)) = 2J' \frac{2}{\sqrt{2\pi}} \overset{!}{=} 1 \Rightarrow 2J' = \frac{2J}{\delta} = \sqrt{\frac{\pi}{2}}$$

6

Lesson 7

For the one-dimensional stochastic motion:

$$\dot{x} = F(x) + \sqrt{\epsilon}\xi$$

with white noise ξ and drift F, the instantons ($\epsilon \to 0$ limit) follow the equation:

$$\ddot{x} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}x} \qquad \text{with} \qquad V_{\text{eff}}(x) = -\frac{F^2(x)}{2}$$

which implies a conservation of the "energy":

$$E = \frac{1}{2}\dot{x}^2 + V_{\text{eff}}(x)$$

Exercise 6.1:

Find the instanton for $F = -\kappa x$ by using the conservation of energy, for initial condition $t_i = 0$, $x_i = 0$, $\dot{x}_i = 0$ and final condition x_0 at $t = 0$.

Solution. By conservation of energy:

$$\mathcal{E} = \frac{1}{2}\dot{x}^2 + V_{\text{eff}}(x) \qquad V_{\text{eff}}(x) = -\frac{x^2\kappa}{2}$$

So we have:

$$\frac{1}{2}\dot{x}^2 = \frac{x^2\kappa^2}{2} \Rightarrow \dot{x} = xk \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = x\kappa \Rightarrow x(t) = x_0 e^{\kappa t}$$

Exercise 6.2:

For $F = -\kappa \sin x$ show that the instanton:

$$x^*(t) = 2\arctan\left(e^{\kappa t}\right)$$

has "energy" $\mathcal{E} = 0$ at every instant t.

Solution. The energy is given by:

$$\mathcal{E} = \frac{1}{2}\dot{x}^2 + V_{\text{eff}}(x) = \frac{1}{2}\dot{x}^2 - \frac{\kappa^2 \sin^2(x)}{2}$$

We substitute the expression for x^* inside $\mathcal{E}$, to compute the energy of the given solution at any instant. We start by computing the $\sin^2$:

$$\begin{aligned}\sin^2 x^*(t) &= \sin^2(2\arctan e^{\kappa t}) = [2\sin\left(\arctan e^{\kappa t}\right)\cos\left(\arctan e^{\kappa t}\right)]^2 = \\ &= 4(\sin^2 \arctan e^{\kappa t})(1 - \sin^2 \arctan e^{\kappa t}) \end{aligned} \tag{6.1}$$

Recall from goniometry:

$$\sin\arctan x = \frac{x}{\sqrt{1+x^2}}$$

And so:

$$\sin^2 \arctan e^{\kappa t} = \frac{e^{2\kappa t}}{1+e^{2\kappa t}}$$

Substituting in (6.1) we get:

$$\begin{aligned}\sin^2 x^*(t) &= \frac{4e^{2\kappa t}}{1+e^{2\kappa t}}\left(1 - \frac{e^{2\kappa t}}{1+e^{2\kappa t}}\right) = \frac{4e^{2\kappa t}}{1+e^{2\kappa t}} \frac{1 + \cancel{e^{2\kappa t}} - \cancel{e^{2\kappa t}}}{1+e^{2\kappa t}} = \\ &= \frac{4e^{2\kappa t}}{(1+e^{2\kappa t})^2}\end{aligned}$$

Then:

$$\mathcal{E} = \frac{1}{2}(\dot{x}^*)^2 - \frac{k^2 \sin^2 x^*}{2} = \frac{1}{2}\left[\frac{4\kappa^2 e^{2\kappa t}}{(1+e^{2\kappa t})^2} - \frac{4\kappa^2 e^{2\kappa t}}{(1+e^{2\kappa t})^2}\right] = 0$$

Exercise 6.3:

Consider a N-dimensional system with $i \leq N$ components. Each component of $\boldsymbol{x} = (x_i)$ follows a stochastic motion:

$$\dot{x}_i = F_i(\boldsymbol{x}) + \sqrt{\epsilon}\xi_i$$

with independent white noises $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$.
By starting from the Euler-Lagrange equation per component, show that the instanton equations become:

$$\ddot{x}_i = \frac{\partial}{\partial x_i}\frac{\|\boldsymbol{F}\|^2}{2} + \sum_{j=1}^N \left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)\dot{x}_j \tag{6.2}$$

where:

$$\|\boldsymbol{F}\|^2 = \sum_{j=1}^N F_i^2$$

Solution We want to minimize the action functional:

$$S[\boldsymbol{x}] = \int_{t_i}^{t_f} L(\boldsymbol{x}, \dot{\boldsymbol{x}})\,\mathrm{d}\tau \qquad L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}\|\dot{\boldsymbol{x}} - \boldsymbol{F}(\boldsymbol{x})\|^2$$

The Euler-Lagrange equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \qquad 1 \leq i \leq N$$

Inserting the expression for L:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\cancel{2}}\cdot\cancel{2}(\dot{x}_i - F_i(\boldsymbol{x})) - \frac{1}{\cancel{2}}\cdot\cancel{2}\sum_{j=1}^N(\dot{x}_j - F_j(\boldsymbol{x}))\left(-\frac{\partial F_j}{\partial x_i}\right) &= \\
= \ddot{x}_i - \sum_{j=1}^N \frac{\partial F_i}{\partial x_j}\dot{x}_j + \sum_{j=1}^N \frac{\partial F_j}{\partial x_i}\dot{x}_j - \sum_{j=1}^N F_j(\boldsymbol{x})\frac{\partial F_j}{\partial x_i} &= 0 \\
\Rightarrow \ddot{x}_i = \underbrace{\sum_{j=1}^N F_j(\boldsymbol{x})\frac{\partial F_j}{\partial x_i}}_{\partial_{x_i}\|\boldsymbol{F}\|^2/2*69} + \sum_{j=1}^N\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)\dot{x}_j & \qquad (6.3)
\end{aligned}$$

Exercise 6.4:

Show that:

$$\mathcal{E} = \frac{1}{2}\|\dot{\boldsymbol{x}}\|^2 + V_{\text{eff}}(\boldsymbol{x}) \qquad V_{\text{eff}}(\boldsymbol{x}) = -\frac{1}{2}\|\boldsymbol{F}\|^2$$

is a constant for the solution of the instanton equations (6.2).

Solution. Differentiating $\mathcal{E}$ wrt time:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \frac{1}{2}2\dot{\boldsymbol{x}}\cdot\ddot{\boldsymbol{x}} - \frac{1}{2}2\boldsymbol{F}\cdot\dot{\boldsymbol{F}} = \\
&= \sum_{i=1}^N \dot{x}_i\,\ddot{x}_i - \sum_{i=1}^N F_i \sum_{j=1}^N \frac{\partial F_i}{\partial x_j}\dot{x}_j = \\
&\underset{(6.3)}{=} \sum_{i=1}^N \dot{x}_i \sum_{j=1}^N \left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right)\dot{x}_j + \sum_{i=1}^N \dot{x}_i \sum_{j=1}^N F_j \frac{\partial F_j}{\partial x_i} - \sum_{i=1}^N F_i \sum_{j=1}^N \frac{\partial F_i}{\partial x_j}\dot{x}_j
\end{aligned}$$

Note that the last two terms cancel out, by exchanging $i \leftrightarrow j$ in the last one. Then we are left with:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \sum_{ij=1}^N \dot{x}_i\dot{x}_j\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right) = \\
&= \sum_{ij=1}^n \dot{x}_i\dot{x}_j\frac{\partial F_i}{\partial x_j} - \sum_{ij=1}^N \dot{x}_j\dot{x}_i\frac{\partial F_j}{\partial x_i} = 0
\end{aligned}$$

Again these last two terms cancel out after substituting $i \leftrightarrow j$ in the last one.